

# Understanding the $q$ -Factors in Quantum Group Symmetry\*

L. C. Biedenharn<sup>†</sup> and K. Srinivasa Rao<sup>\*\*</sup>

Department of Physics, Duke University, Durham, North Carolina 27706, USA

Z. Naturforsch. **52a**, 59–62 (1997)

A characteristic feature of quantum groups is the occurrence of  $q$ -factors (factors of the form  $q^k$ ,  $k \in \mathbb{R}$ ), which implement braiding symmetry. We show how the  $q$ -factors in matrix elements of elementary  $q$ -tensor operators (for all  $U_q(n)$ ) may be evaluated, without explicit calculation, directly from structural symmetry properties.

## Introduction

George Sudarshan is one of the pioneers and leading practitioners in the application of symmetry techniques to theoretical physics, and accordingly we feel that the remarks to follow on the newest development in symmetry – quantum groups – will appeal to him and hence should be appropriate for this colloquium in his honor.

## 1. Quantum Groups

What are quantum groups? The precise definition [1] is rather forbidding and, in fact the quickest way to understand this new structure is by example. Let us consider the prototypical example: the quantum group  $SU_q(2)$ .

$SU_q(2)$  is generated by the three operators  $J_\pm^q, J_z^q$  satisfying the commutation relations

$$[J_z^q, J_\pm^q] = \pm J_\pm^q, \quad (1.1)$$

$$[J_+^q, J_-^q] = \frac{q^{J_z^q} - q^{-J_z^q}}{q^{1/2} - q^{-1/2}}, \quad q \in \mathbb{R}^+. \quad (1.2)$$

These defining relations for  $SU_q(2)$  differ from those of ordinary angular momentum ( $SU(2)$ ) in two ways:

- (a) The commutator in (1.2) is not  $2J_z$  as usual, but an infinite series (for generic  $q$ ) involving *all odd powers*:

\* Presented at a Workshop in honor of E. C. G. Sudarshan's contributions to Theoretical Physics, held at the University of Texas in Austin, September 15–17, 1991.

\*\* Permanent address: The Institute of Mathematical Sciences, Madras-600113, India.

<sup>†</sup> Deceased; last address: Department of Physics, Center for Particle Theory, The University of Austin, Austin TX 78712, USA.

Reprint requests to Prof. Dr. K. S. Rao.

ers:  $(J_z^q)^1, (J_z^q)^3, \dots$ . Each such power is a linearly independent operator in the enveloping algebra; accordingly, the Lie algebra of  $SU_q(2)$  is *not* of finite dimension.

- (b) For  $q \rightarrow 1$ , the right hand side of (1.2)  $\rightarrow 2J_z$ . Thus we recover in the limit the usual Lie algebra of  $SU(2)$ .

The differences noted in (a) and (b) are expressed by saying that the quantum group  $SU_q(2)$  is a *deformation* of the enveloping algebra of  $SU(2)$ . Let us introduce a notation for the “ $q$ -integers” that play a role in quantum groups. Define the  $q$ -integer  $[n]_q$  by

$$[n]_q \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad = \underbrace{q^{(n-1)/2} + q^{(n-3)/2} + \dots + q^{-(n-1)/2}}_{n \text{ terms}}, \quad n \in \mathbb{Z}. \quad (1.3)$$

These  $q$ -integers,  $[n]_q$ , obey the rule:  $[-n]_q = (-1)[n]_q$ , with  $[0]_q = 0$  and  $[1]_q = 1$ . Note that  $[n]_q = [n]_{q^{-1}}$ , so that the defining relations (1.1) and (1.2) are invariant to  $q \leftrightarrow q^{-1}$ . (This – with the use of steps of *unity* for powers of  $q$  in (1.3) – accounts for the convention using  $q^{1/2}$  in (1.3).)

Anticipating that  $2J_z^q$  has integer eigenvalues, we can then write (1.2) in an equivalent operator form:

$$[J_+, J_-] = [2J_z]_q. \quad (1.4)$$

In order to understand the meaning of (1.1) and (1.2) it is natural to look for representations of the operators  $J_\pm, J_z$  as finite-dimensional matrices. It is remarkable that (for *generic*  $q$ ) all irreducible representations of  $SU_q(2)$  are finite-dimensional and moreover, each such irrep is a deformation of an irrep of  $SU(2)$  having precisely the same dimension.

This basic results – which can be easily proved by a  $q$ -boson realization [2] – is a special case of the

0932-0784 / 97 / 0100-0059 \$ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72072 Tübingen



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

*Lusztig-Rosso theorem* [3, 4] which asserts that for generic  $q$ , the dimensions (of the unitary irreps of a  $q$ -deformation of a compact classical Lie group) are deformation invariants.

The symmetry structure in quantum groups is accordingly very close to that of the original Lie group, but this does not mean there are no significant changes! For, recall that the *commutation relations* (and hence the underlying physics) *have changed* (“deformed”) and moreover in a Hamiltonian theory having quantum group symmetry the energy level spacing (but not degeneracy) can change. Such structural deformations have been an important source of new physics in the past; one need only recall that quantum mechanics can be viewed as a deformation of classical mechanics, and similarly special relativity can be taken in some sense as a deformation of Galilean relativity.

## 2. Quantum Groups as Hopf Algebras

Drinfeld [1] – who gave the first abstract characterization of quantum groups – defined a quantum group as *the spectrum of a (not necessarily commutative) Hopf algebra*. Hopf algebras must surely be considered as *terra incognita* to most physicists, but we claim that, in fact, the ingredients of Hopf algebra are not only natural in quantum physics but familiar and essential (the cliché on Ms. Jourdain and prose springs to mind).

A Hopf algebra is a bi-algebra, that is to say, one takes an associative algebra  $\mathbf{A}$  (with a unit  $\mathbf{1}$ ) over a field  $k$  having

$$\text{multiplication: } \mathbf{A} \otimes \mathbf{A} \xrightarrow{m} \mathbf{A} \quad (2.1)$$

$$\text{and unit: } k \rightarrow \mathbf{A}, \quad (2.2)$$

(given by  $k \rightarrow k\mathbf{1}$ ), and then to get a bi-algebra one adjoins additional operations that *reverse the arrows*:

$$\text{co-multiplication: } \mathbf{A} \xrightarrow{\Delta} \mathbf{A} \otimes \mathbf{A} \quad (2.3)$$

$$\text{and co-unit: } \mathbf{A} \xrightarrow{\epsilon} k. \quad (2.4)$$

(There are additional, mainly compatibility, laws, but we omit these for simplicity.)

The distinguishing characteristic here is the *co-multiplication*  $\Delta$  – what is this physically? To answer this, consider the angular momentum operator  $\mathbf{J}$ . In both quantum and classical mechanics one can *add* angular momenta: given two angular momenta  $\mathbf{J}^{(1)}$  and  $\mathbf{J}^{(2)}$ ,

we can construct the sum,  $\mathbf{J}_{\text{total}}$ :

$$\mathbf{J}_{\text{total}} \equiv \mathbf{J}^{(1)} + \mathbf{J}^{(2)}. \quad (2.5)$$

For independent (commuting) quantal angular momenta,  $\mathbf{J}_{\text{total}}$  obeys the same commutation rules as each of the  $\mathbf{J}^{(i)}$ .

More formally,  $\mathbf{J}^{(1)}$  acts, say, on the space of kets  $|\psi\rangle_1$ , and  $\mathbf{J}^{(2)}$  on the independent space  $|\varphi\rangle_2$  in the tensor product space  $|\psi\rangle_1 \otimes |\varphi\rangle_2$ . Thus, to be more precise, we should write

$$\mathbf{J}_{\text{total}} \equiv \mathbf{J}^{(1)} \otimes \mathbf{1}^{(2)} + \mathbf{1}^{(1)} \otimes \mathbf{J}^{(2)}. \quad (2.6)$$

If we now recognize that these operators, including  $\mathbf{1}$ , each belong to a formal algebra of operators – call it  $\mathbf{A}$  –, then we may write (2.6) as a co-multiplication:

$$\mathbf{J}_{\text{total}} \equiv \Delta(\mathbf{J}) = \mathbf{J} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J}, \quad (2.7)$$

that is, the operation denoted  $\Delta$  takes an element of  $\mathbf{A}$  (here  $\mathbf{J}$ ) and carries it into an element of the tensor product:  $\mathbf{A} \otimes \mathbf{A}$  – this is, of course, the meaning of a co-multiplication. (We chose, for simplicity, independent angular momenta, but in the usual realization by first order derivatives it is correct to take the *same* angular momentum in the addition law and hence the co-multiplication.)

We conclude: *the usual addition of quantal angular momentum defines a commutative co-multiplication*. Hence angular momentum theory in quantum mechanics has a natural Hopf algebra structure.

For the quantum group  $\text{SU}_q(2)$ , the co-multiplication takes one of two forms: either

$$\Delta(J_{\pm}) = q^{-J_z/2} \otimes J_{\pm} + J_{\pm} \otimes q^{J_z/2}, \quad (2.8a)$$

$$\Delta(J_z) = \mathbf{1} \otimes J_z + J_z \otimes \mathbf{1}, \quad (2.8b)$$

or the above form with  $q$ -replaced by  $q^{-1}$ . (Note that in (2.8a) the two terms on the RHS are not symmetric.)

The real significance of quantum groups is therefore that one has a *non-commutative* co-multiplication. This means (using our analogy with angular momentum) that:

- (a) the commutation relations are *changed*, and
- (b) the addition of  $q$ -angular momentum is not commutative, but depends on *order*. [At first glance, it may appear that inherent symmetry of three-space is *broken* (the  $z$ -direction is singled out both in the commutation relations and in the co-multiplication), but a further analysis shows that this conclusion is not correct since the *degeneracy structure* is not changed.]

These are major changes, and to accord with this, particle exchange symmetry is no longer given by the symmetric group (as it would be for a co-commutative Hopf algebra) but by a Hecke algebra [5] (related to the braid group) with its unusual statistics. It is remarkable that these new developments in symmetry were obtained, not by philosophical speculation, but by induction from actual solvable (two-dimensional) physical models [5, 6].

### 3. Some Consequences of the Lusztig-Rosso Theorem

We have noted above that the Lusztig-Rosso theorem implies the invariance of the irrep dimensions under deformation. This has very important consequences when we extend this theorem to  $q$ -tensor operators, (the  $q$ -extension of the Wigner-Eckart theorem [7]).

Consider, for example, the fundamental (spin- $\frac{1}{2}$ )  $q$ -tensor operators in  $SU_q(2)$ . Since the irrep structure is invariant under deformation, it follows that the spin- $\frac{1}{2}$   $q$ -tensor operators – just as for ordinary angular momentum theory – necessarily induce either the irrep label shift:  $j \rightarrow j + \frac{1}{2}$  or the shift:  $j \rightarrow j - \frac{1}{2}$ .

The structural properties of tensor operators for the usual angular momentum theory [8], show that the matrix elements for all spin- $\frac{1}{2}$  unit tensor operators are completely determined (to within  $\pm$  phases) by the *structural zeroes* (linear factors) implied by group theoretical constraints on irrep label shifts. An example will make this concept clear. Consider the spin- $\frac{1}{2}$  operator having  $J = \frac{1}{2}$  and  $M = \frac{1}{2}$  which induces the shift  $j \rightarrow j - \frac{1}{2}$ . The selection rules imply that this operator must have a vanishing matrix element when operating on the irrep vector  $|j, j\rangle$ , having angular momentum  $j$  and  $z$ -component ( $m$ ) also equal to  $j$ . [This is because the would-be final state  $(j, m) = (j - \frac{1}{2}, j + \frac{1}{2})$  does not exist.] This constraint implies that the matrix element (squared) *must contain the linear factor*  $(j - m)$ , as can be verified from the usual tables.

It is a well-known, but nonetheless remarkable, result [8, 9] that such structural considerations – augmented by permutational symmetry in the proper (hook) labels – *completely determine* (to within  $\pm$  signs, set by phase conventions) *all matrix elements of all elementary tensor operators for all*  $SU(n)$ . [An elementary operator is one whose irrep labels (Young frame) consist of only 1's and 0's.]

There is an extension [10] of this result to the  $q$ -tensor operators of  $U_q(n)$ : as shown by the  $q$ -boson real-

ization, *the basic change is the replacement of each linear factor ( $f$ ) by the corresponding  $q$ -integer  $[f]$ .*

Since the  $(\pm)$  phase is necessarily invariant to (continuous) changes in the parameter  $q$ , this standard result determines the matrix elements the matrix elements of all elementary  $q$ -tensor operators in all  $U_q(n)$  *up to a possible  $q$ -factor (that is, a factor  $q^a$ ) multiplying any given matrix element.*

*To summarize: structural properties of the  $q$ -tensor operators implied by the Lusztig-Rosso theorem suffice to determine the matrix elements of all elementary  $q$ -tensor operators in all  $U_q(n)$ , up to possible  $q$ -factors.*

### 4. How to Determine the $q$ -Factors from Structural Properties

In the previous sections, we have indicated that one can determine directly the matrix elements of all elementary tensor operators (which all have monomial matrix elements) *up to unknown  $q$ -factors* (factors of the form  $q^a$ ). Our problem is now to determine these factors without appeal to explicit calculation.

Studies of the structure of angular momentum theory  $SU(2)$  and extensions to the  $SU(n)$  group have shown [8] that the  $(3n-j)$  coefficients are structurally interrelated by a sequence of limit operations. The best known such result [11] relates, in  $SU(2)$ , the  $(6-j)$  symbol to the  $(3-j)$  coefficients (unit tensor operator matrix elements); that is

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} \rightarrow \left( \begin{matrix} a & b & e \\ c-f & f-d & d-c \end{matrix} \right) \quad (4.1)$$

in the limit  $c, d, f \rightarrow \infty$  with  $e-f, f-d, d-e$  finite.

This structural result is known to be valid [10] for the  $q$ -extension of the  $(6-j)$  and  $(3-j)$  symbols [denoted as  $q(6-j)$  and  $q(3-j)$  symbols].

At first glance, this result, elegant though it may be, seems of little use in solving our problem: the  $q(6-j)$  coefficients are at least as difficult to determine explicitly as the  $q(3-j)$  coefficients!

If one examines, however, the defining relations for  $SU_q(2)$ , (1.1) and (1.4), one sees that the Lie algebra structure is invariant to the replacement  $q \rightarrow q^{-1}$ . It is the co-algebra structure (the co-multiplication  $\Delta$ ) which distinguishes  $q$  from  $q^{-1}$ , and it is this co-multiplication structure that directly determines the  $q(3-j)$  coefficients, which are in consequence *not* invariant to  $q \rightarrow q^{-1}$ .

The  $q$ -(6- $j$ ) coefficients are defined in terms of four  $q$ -(3- $j$ ) coefficients, and it is from this explicit definition that one discovers the remarkable fact that *the  $q$ -(6- $j$ ) symbols are invariant to the symmetry  $q \rightarrow q^{-1}$ .*

An intuitive way to understand this symmetry comes from representing the  $q$ -(6- $j$ ) as a tetrahedron. If we orient the four faces of the tetrahedron (corresponding to ordering the tensor products), we see that: (a) each  $q$ -angular momentum enters *twice* and (b) the “flow” (order) for each  $q$ -angular momentum occurs in *both directions*. (There is, in fact no consistent way to define a *unique* flow for all  $q$ -angular momenta involved.) This geometry property strongly suggests that the  $q$ -(6- $j$ ) is invariant to orientation (equivalently,  $q \rightarrow q^{-1}$ ), which is, of course, true. (Alternatively, one can argue that the  $q$ -(6- $j$ ) coefficients are functions of invariants, the  $q$ -invariant obeys the symmetry:  $q \rightarrow q^{-1}$ .)

By using this limit relation, (4.1), we can accordingly determine the  $q$ -factor entering in the explicit  $q$ -(3- $j$ ) matrix elements, and thereby complete our program of determining all matrix elements of all elementary tensor operators *directly from structural information*. (One might be temporarily disturbed that a symmetrical starting point – the  $q$ -(6- $j$ ) – gives an unsymmetrical result – the  $q$ -(3- $j$ ). However, the symmetry breaking is actually a property of the limit which distinguishes  $q^f$  from  $q^{-f}$  for  $q$  real and  $f$  large.)

It remains only to remark that the pattern calculus [9, 10] (which encodes this structural information for  $U_q(n)$ ) makes this approach a very practical one. The relevant  $q$ -(6- $j$ ) coefficients are *monomials*, and the limit involves only ratios of linear factors ( $q$ -integers).

## 5. Concluding Remarks

The  $q$ -extension of the classical Lie groups, that is, the construction of the associated quantum groups, involves for physical applications the  $q$ -extension of the (generalized) hypergeometric functions. (Thus, for example, the  $SU_q(2)$  (3- $j$ ) coefficients [12] are  ${}_3\phi_2$  basic hypergeometric functions.) Such an extension requires that one manipulates algebraic expressions involving the  $q$ -integers,  $[n]$ . This  $q$ -structure is extremely unwieldy for calculations, not least because the  $q$ -integers do not admit a ring structure.

To appreciate the difficulties this poses consider *adding* two  $q$ -integers:  $[m]$  and  $[n]$ . This addition is well-defined as an algebraic expression in  $q$ , but the sum is not necessarily a  $q$ -integer. To “add” two  $q$ -integers, one must use

$$[m] \oplus [n] \equiv q^{-n/2} [m] + q^{m/2} [n] = [m + n]. \quad (5.1)$$

It is easily seen (noting the properties given after (1.3)) that the  $\oplus$  operation defines an Abelian group isomorphic to  $\mathbb{Z}$ , the group of ordinary integers. This structure indicates that “ $q$ -factors” will necessarily play a very essential role in actual calculations with quantum group symmetry.

From this point of view, it is something of an ‘algebraic miracle’ that the orthonormality properties of the  $q$ -(3  $n$ - $j$ ) coefficients are actually preserved. Quantum groups have many such improbable and mysterious features, which are a strong hint that there is something important hidden here!

## Acknowledgement

Supported in part by the National Science Foundation, No. PHY-9008007.

- [1] V. G. Drinfeld, Quantum Groups. ICM Proceedings, Berkeley, 798–820 (1996).
- [2] L. C. Biedenharn, J. Phys. A: Math. Gen. **22** L, 873 (1989); A. J. Macfarlane, J. Phys. A: Math. Gen. **22**, 4581 (1989).
- [3] G. Lusztig, Adv. in Math. **70**, 237 (1988).
- [4] M. Rosso, Commun. Math. Phys. **117**, 581 (1988).
- [5] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985).
- [6] E. Sykkanin, L. Takhtajan, and L. Faddeev, Teor. Math. Phys. **40**, 194 (1979) (In Russian); P. Kulish and N. Reshetikhin, Zap. nauch. seminarov LOMI **101**, 101 (1981) (in Russian).
- [7] L. C. Biedenharn and M. Tarlini, Lett. Math. Phys. **20**, 271 (1990).
- [8] L. C. Biedenharn and J. D. Louck, Angular Momentum in Quantum Physics, Theory and Application, Encyclopedia of Mathematics and its Applications, Vol. 8, Reading Massachusetts: Addison-Wesley, 1981.
- [9] L. C. Biedenharn and J. D. Louck, Comm. Math. Phys. **8**, 89 (1968).
- [10] L. C. Biedenharn, Quantum Groups, Proceedings of the 8th International Workshop of Mathematical Physics, July 1989, Lecture Notes in Physics **370**, 67 (1990).
- [11] L. C. Biedenharn, J. Math. and Phys. (MIT) **31**, 287 (1953).
- [12] V. Rajeswari and K. Srinivasa Rao, J. Phys. A: Math. Gen. **24**, 3761 (1991).